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6. AUTHORS

Nickolas Vlahopoulos

7. PERFORMING ORGANIZATION NAMES AND ADDRESSES

University of Michigan - Ann Arbor
Regents of the University of Michigan
3003 S. State St
Ann Arbor, MI 48109 -1274

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14. ABSTRACT

Two different types of simulations are required for analyzing Army vehicles when they are subjected to explosive/impact loads: (a) Survivability analysis, when loads generate damage on the vehicle structure by causing permanent deformation; (b) Shock analysis, when the vehicle structure remains within the elastic region. Shock analysis ensures that vibration levels must remain low at locations where electronic equipment is mounted. Due to short duration of the load the high frequency response is important for shock analysis (i.e. 500Hz – 10,000Hz).

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Report Title

Modeling High Frequency Vibration in Composites Using an Energy Finite Element Method for Shock Analysis of Lightweight Army Vehicles

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Two different types of simulations are required for analyzing Army vehicles when they are subjected to explosive/impact loads: (a) Survivability analysis, when loads generate damage on the vehicle structure by causing permanent deformation; (b) Shock analysis, when the vehicle structure remains within the elastic region. Shock analysis ensures that vibration levels must remain low at locations where electronic equipment is mounted. Due to short duration of the load the high frequency response is important for shock analysis (i.e. 500Hz – 10,000Hz). Since the shock loads comprise one of the sets of design loads for any Army vehicle, it is important to perform a shock simulation efficiently enough in order to make design decisions. Such simulations will allow to access the survivability of the equipment and therefore of the vehicle, and they will allow to incorporate design changes in order to achieve desirable shock spectra at mounting locations. Computing efficiently the high frequency response of composite structures is the objective of this research.

List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

Number of Papers published in peer-reviewed journals: 0.00

(b) Papers published in non-peer-reviewed journals or in conference proceedings (N/A for none)

Number of Papers published in non peer-reviewed journals: 0.00

(c) Presentations

Number of Presentations: 0.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts): 0

Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Peer-Reviewed Conference Proceeding publications (other than abstracts): 0

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Number of Manuscripts: 0.00

Patents Submitted

Patents Awarded

Awards

Xiaoyan Yan, the graduate student supported during the first year by this project won the outstanding graduate student award for the NA&ME Department for 2008. Professor Vlahopoulos, who is the PI of this project, received the 2011 NA&ME Departmental Faculty award.

Graduate Students

NAME	PERCENT_SUPPORTED
Sungmin Lee	0.50
Xiaoyan Yan	0.50
FTE Equivalent:	**1.00**
Total Number:	**2**

Names of Post Doctorates

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Names of Faculty Supported

NAME	PERCENT_SUPPORTED	National Academy Member
Nickolas Vlahopoulos	0.10	No
FTE Equivalent:	**0.10**	
Total Number:	**1**	

Names of Under Graduate students supported

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Student Metrics

This section only applies to graduating undergraduates supported by this agreement in this reporting period

The number of undergraduates funded by this agreement who graduated during this period: 0.00

The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields:...... 0.00

The number of undergraduates funded by your agreement who graduated during this period and will continue to pursue a graduate or Ph.D. degree in science, mathematics, engineering, or technology fields:...... 0.00

Number of graduating undergraduates who achieved a 3.5 GPA to 4.0 (4.0 max scale):...... 0.00

Number of graduating undergraduates funded by a DoD funded Center of Excellence grant for Education, Research and Engineering:...... 0.00

The number of undergraduates funded by your agreement who graduated during this period and intend to work for the Department of Defense 0.00

The number of undergraduates funded by your agreement who graduated during this period and will receive scholarships or fellowships for further studies in science, mathematics, engineering or technology fields:...... 0.00

Names of Personnel receiving masters degrees

NAME

Total Number:

Names of personnel receiving PHDs

NAME
Xiaoyan Yan

Total Number: **1**

Names of other research staff

NAME	PERCENT_SUPPORTED

FTE Equivalent:

Total Number:

Sub Contractors (DD882)

Inventions (DD882)

Scientific Progress

Technology Transfer

Final Report

Modeling High Frequency Vibration in Composites Using an Energy Finite Element Method for Shock Analysis of Lightweight Army Vehicles

(Proposal No. 51087EG)

Nickolas Vlahopoulos
Professor
Department of Naval Architecture and Marine Engineering
Department of Mechanical Engineering
University of Michigan, Ann Arbor, MI 48109

I. Research Objectives

- Two different types of simulations are required for analyzing Army vehicles when they are subjected to explosive/impact loads.

(a) Survivability analysis; when loads generate damage on the vehicle structure by causing permanent deformation (typical results are presented in Figure 1).

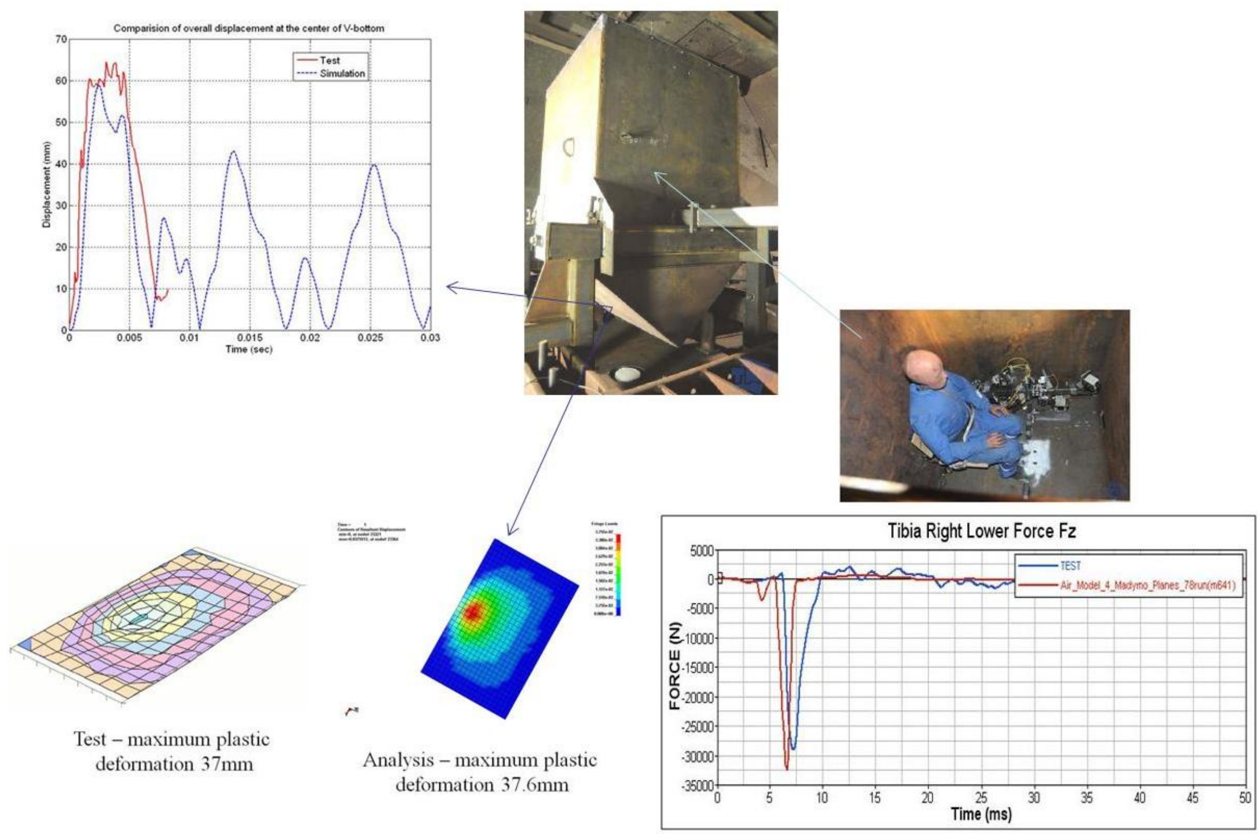
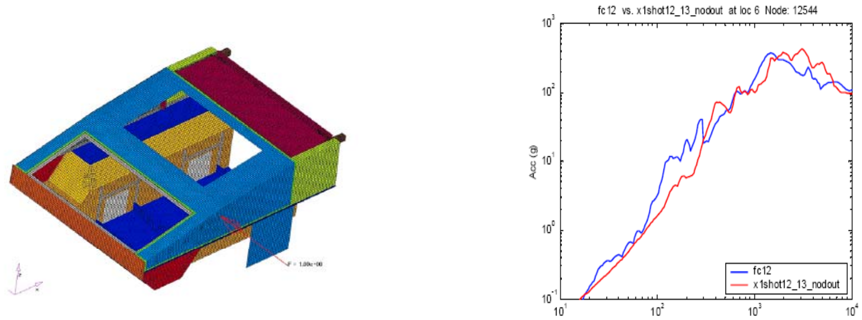


Figure 1. Survivability analysis for vehicle/occupant response to explosive threat [N. Vlahopoulos, G. Zhang, "Validation of a Simulation Process for Assessing the Response of a Vehicle and its Occupants to an Explosive Threat," 27th Army Science Conference]

(b) Shock analysis; when the vehicle structure remains within the elastic region. Shock analysis ensures that vibration levels must remain low at locations where electronic equipment is mounted. Due to short duration of the load the high frequency response is important for shock analysis (i.e. 500Hz – 10,000Hz). Typical shock analysis results are presented in Figure 2.



(a) (b)

Figure 2. (a) Plate assembly structure analyzed for shock; (b) response spectra at a point [R. D. Hampton, N. S. Wiedenman, T. H. Li, "Analytical Determination of Shock Response Spectra," 11th ARL/USMA Technical Symposium]

- In an effort to make Army vehicles more lightweight, composite materials can be used for their construction. ***Efficient shock vibration analysis of composite vehicle comprises the objective of this research effort***.

II. Approach

- Traditionally, Finite Element Analysis (FEA) solutions are used in order to compute the vibration of a structure due to blast/shock. These conventional FEA methods compute the displacement of the vibration at each point of the structure with respect to time. At high frequencies the wavelengths of the vibration are small and a large number of nodes and elements are required in a FEA model in order to capture correctly the shape of the vibration due to the small wavelengths. The large number of elements in the FEA model result in requirements for extremely high computational resources. Thus, high frequency shock analyses for an entire vehicle using conventional FEA methods are very expensive and even infeasible after a certain frequency.

- The Energy Finite Element Analysis (EFEA) formulation is based on using the space averaged over a wavelength and time averaged over a period energy density (energy density) as a primary variable. The governing differential equations of the EFEA formulation are developed for the energy density. The energy density varies exponentially in space, thus only a small number of elements is required in order to capture the high frequency vibration. Figure 3 depicts graphically the differences in the primary variables between conventional FEA and EFEA methods. Once the governing differential equations are identified, the finite element approach can be employed for generating the corresponding numerical system of equations and the element matrices.

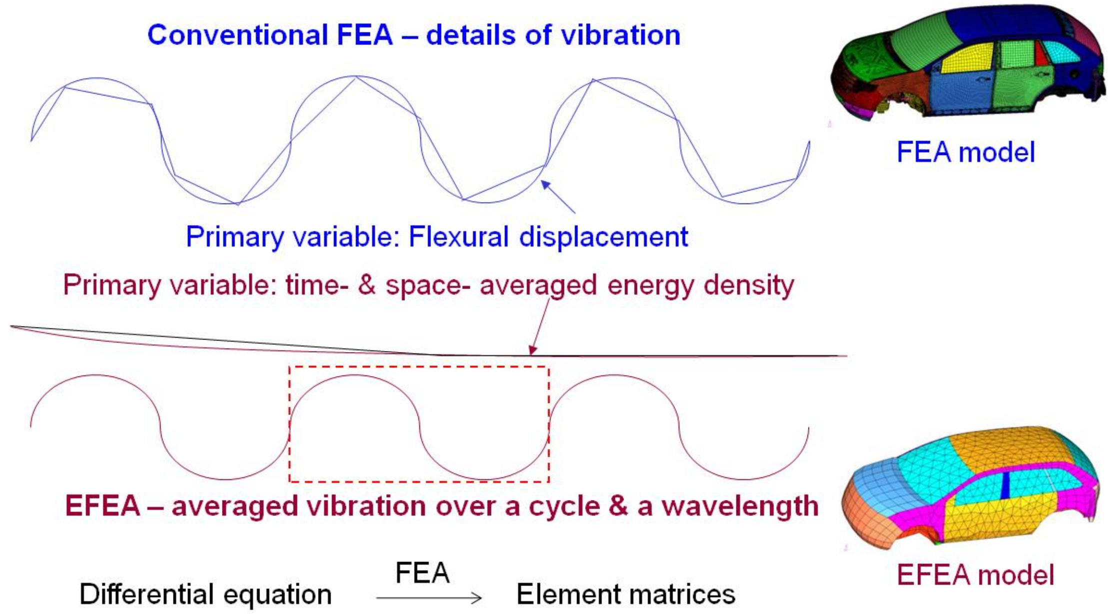


Figure 3. Graphical representation of the differences between conventional FEA and EFEA and the main reason for efficiency in EFEA calculations due to the small number of elements

- The main challenges in the EFEA arise when different structural components are connected together. Unlike the conventional FEA, where the primary variable remains continuous at the connection, in the EFEA the primary variable becomes discontinuous. This situation is represented graphically in Figure 4. Specialized joint formulations must be developed for connecting the element level matrices across a joint based on power balance equations.

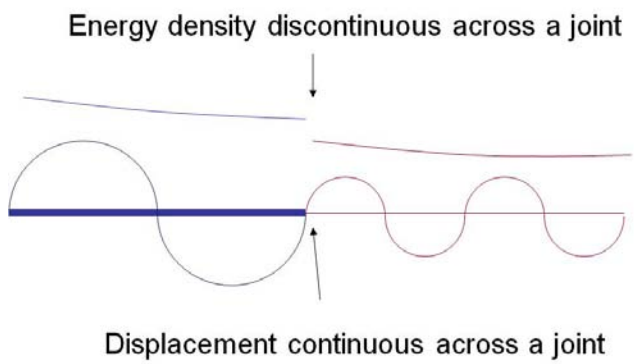


Figure 4. Graphical representation of the challenges in the EFEA at connections between components

- In order to develop an EFEA formulation for composite structures two main new developments must be completed and implemented into software:
  (i) Development of the governing differential equations for composite materials and the associated element matrices.
  (ii) Development of joint formulations for the connections between composite components.
  These two tasks comprise the main focus of this research project.

III. Significance

- In a continuous effort to reduce fuel consumption of Army vehicles, light weight vehicle structures with increased functionality must be designed. Since the shock loads comprise one of the sets of design loads for any Army vehicle, it is important to perform a shock simulation efficiently enough in order to make design decisions. Such simulations will allow to access the survivability of the equipment and therefore of the vehicle, and they will allow to incorporate design changes in order to achieve desirable shock spectra at mounting locations.

- Currently, typical computational times based on conventional finite element methods require two days per impact load (Figure 2). From the response spectra it can be observed that the majority of the shock energy appears in the frequency range 1,000Hz – 10,000Hz.

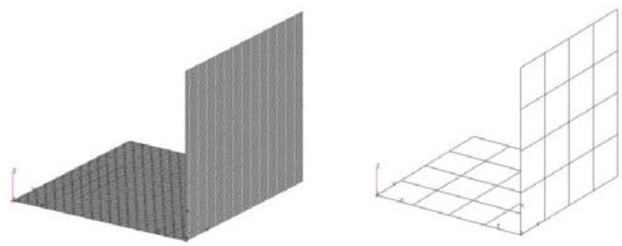
- The EFEA is an established simulation method for metallic structures (Figure 5). In this project research has been conducted for enabling the utilization of the EFEA for analyzing composite structures.

Figure 5. EFEA applications for metallic structures

IV. Accomplishments in the current grant

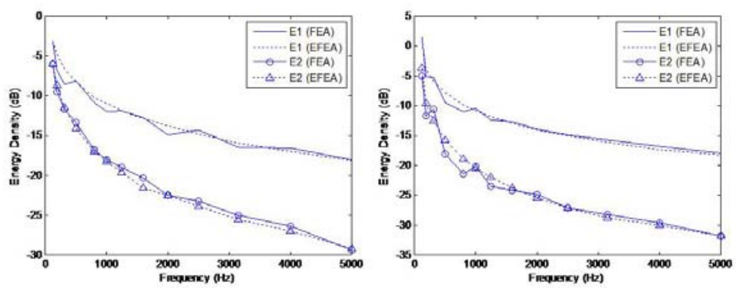
- Development of the governing differential equations and element level matrices for composite materials. Appendix A contains detailed information about the derivation of the element level EFEA matrices for composite materials.
- Development of joint formulations for the connections between composite components. Appendix B contains technical information about the computation of the power transfer coefficients and of the joint matrices.
- Use the element matrices and the joint matrices for creating the system of equations that represents complex composite structures

- Gradual validation through comparison with conventional finite element solutions for simple structures (Figures 6 and 7) and for a complex structure (Figures 8 and 9) similar to the one tested under explosive loads in Figure 1. For the composite structure with a double floor and outer V shaped bottom presented in Figure 8 the excitation is applied in the middle of the outer V shaped floor.



The model in conventional FEA has 12,800 elements.
The model in EFEA has only 32 elements.

Figure 6. Conventional FEA and EFEA models for two composite panels



Case 1 Case 2

Computation time: 6.5 hours vs. 5 minutes

Figure 7. Comparison between FEA and EFEA results for two different configurations of connecting composite panels one and two along two different edges of panel 1

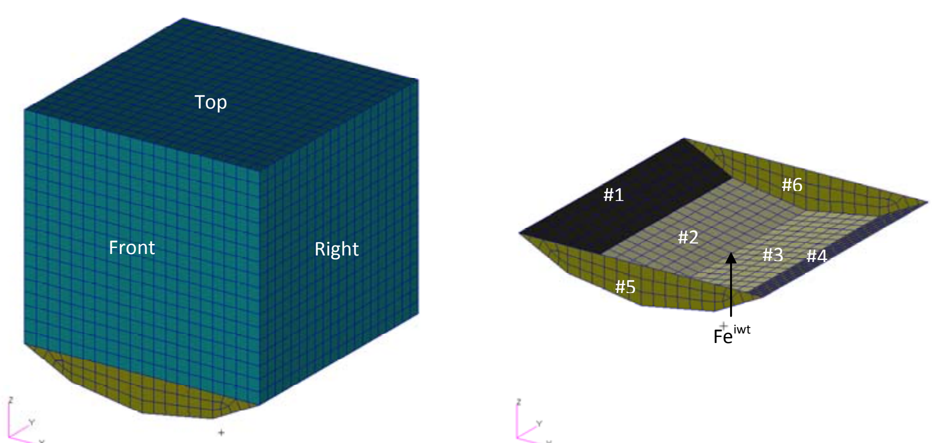


Figure 8. All composite structure with a double floor and outer V shaped bottom; EFEA model

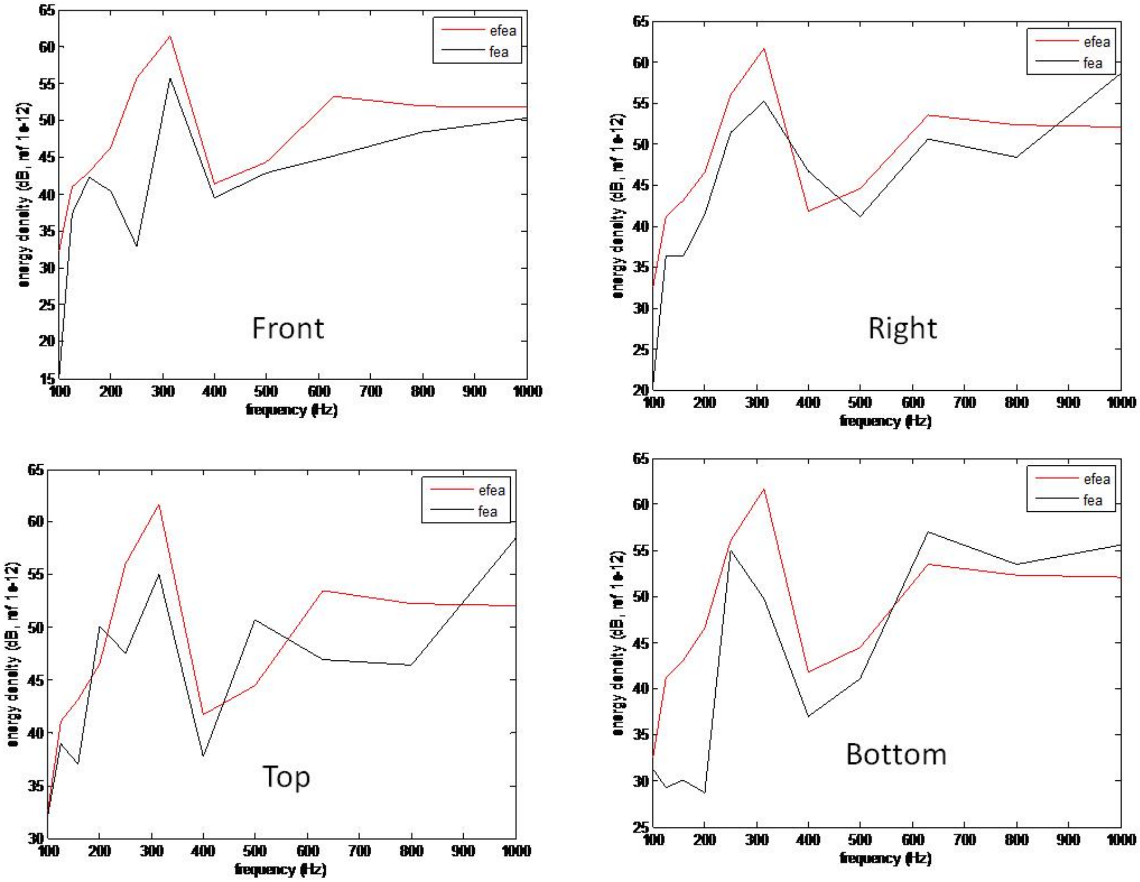


Figure 9. Comparison between conventional FEA and EFEA results for the response of the various panels

Figure 9 presents the correlation between the conventional FEA model (established computational method) and the new EFEA method for modeling composite structures. The agreement between the two solutions in terms of capturing the frequency dependency of the vibration at the different parts of the structure, along with the magnitude of the response demonstrate the validity of using the EFEA for modeling the transfer of vibrational energy in composite structures. The analysis was terminated at 1,000Hz because this is the upper limit of validity of the conventional FEA model used in this study. There is no upper frequency limit in the EFEA model. In this case study the EFEA simulations are 25 times faster compared to the FEA for the same desktop computer.

V. Technology Transfer

Continuous interactions is in place with Farzad Rostam –Abedi and Kris Bishnoi from the survivability group of TARDEC. Visits and seminars on this topic have been given at TARDEC. Several interactions have also taken place with Matt Castanier from the CASI group of TARDEC.

VII. Awards and Honors

Xiaoyan Yan, the graduate student supported during the first year by this project won the outstanding graduate student award for the NA&ME Department for 2008. Professor Vlahopoulos, who is the PI of this project, received the 2011 NA&ME Departmental Faculty award.

VIII. Leveraged Funding

A project was funded by NASA Langley for extending the composite EFEA developments for rotorcraft applications, and for validating the method through comparison to test data.

Appendix A – Derivation of Element Level Matrices for EFEA composites

A Spectral Finite Element Method (SFEM) is used for evaluating the group speed c_g^* and the damping η^* due to the effectiveness of SFEM for taking into account the layer-wise composition of composite panels along with the shear deformation effects. The c_g^* and η^* values computed by the SFEM are used in the EFEA formulation for deriving the element level matrices of the composite materials. The basic steps of the SFEM approach are presented here. Figure A.1. depicts a plane wave propagating in the positive x direction with frequency ω and wavenumber k in a multilayer composite plate. The x direction comprises a reference for orienting the various layers. The through-thickness discretization in the z-axis along with the assumption of the harmonic wave motion in the x direction results in the following form of displacement field, $\mathbf{u}^T = [u, v, w]$ at any point, (x, y, z) within the plate,

$$\mathbf{u}(x, z, t) = \mathbf{N}(z)\hat{\mathbf{u}}e^{\mathrm{i}(\omega t - kx)} \tag{1}$$

where $\mathbf{N}(z)$ is a matrix of shape functions and $\hat{\mathbf{u}}$ is a vector of the nodal degrees of freedom of the form:

$$\hat{\mathbf{u}} = [u_1, v_1, w_1 \quad u_2, v_2, w_2 \quad \cdots \quad u_{N_e+1}, v_{N_e+1}, w_{N_e+1}]^T \tag{2}$$

where N_e is the total number of linear finite elements through the thickness. It is noted that any of the displacement components in $\hat{\mathbf{u}}$ may be complex numbers.

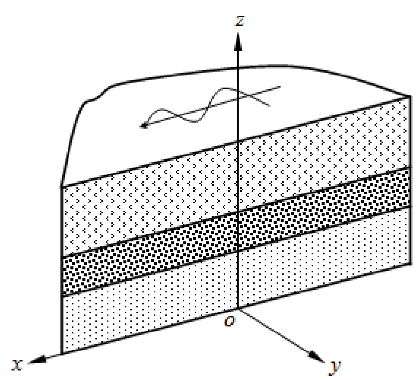


Figure A.1: A plane wave propagation in a multilayer composite panel

The time-averaged total kinetic and potential energies, $\langle T \rangle$ and $\langle V \rangle$ are given by

$$\langle T \rangle = \frac{\omega^2}{4}\int_\Omega \rho \mathbf{u}^H \mathbf{u} d\Omega; \langle V \rangle = \frac{1}{4}\int_\Omega \boldsymbol{\sigma}^H \boldsymbol{\varepsilon} d\Omega \tag{3}$$

where ^{H} stands for the Hermitian transpose. Substitution of appropriate stress-strain and strain-displacement relations and the displacement field equation (1) into equation (3) results in:

$$\langle T \rangle = \hat{\mathbf{u}}^H \omega^2 \mathbf{M}\hat{\mathbf{u}}; \langle V \rangle = \hat{\mathbf{u}}^H \mathbf{K}\hat{\mathbf{u}} \tag{4}$$

where **K** and **M** are, respectively, the stiffness and mass matrices. The replacement of the spatial derivatives with respect to x and z by $-\mathrm{i}k$ and $d\mathbf{N}/dz$ can yield the expressions for the stiffness and mass matrices. Using Hamilton's principle results in:

$$[\mathbf{K}(k) - \omega^2\mathbf{M}]\hat{\mathbf{u}} = \mathbf{0} \tag{5}$$

Since $\mathbf{K} = \mathbf{K}(k)$, for a prescribed circular frequency ω, equation (5) can be solved for the eigenvalues ω^2 and eigenvectors $\hat{\mathbf{u}}$. The hysteretic damping model can be applied to each layer to yield the following form for the total time-averaged energy loss associated with an arbitrary wave type,

$$\langle \underline{\pi}_{diss} \rangle = \sum_{l=1}^{N_e} \eta_l \omega \langle e \rangle_l \tag{6}$$

where η_l and $\langle e \rangle_l$ are, respectively, the structural loss factor and energy density of the lth layer of multi-layered composites. Since $\langle e \rangle = \langle T \rangle + \langle V \rangle = 2\langle V \rangle$, the damping loss factor associated with each propagating wave can be expressed as

$$\eta = \frac{\sum_{l=1}^{N_e} \eta_l \hat{\mathbf{u}}^H \mathbf{K}^{(l)} \hat{\mathbf{u}}}{\hat{\mathbf{u}}^H \mathbf{K} \hat{\mathbf{u}}} \tag{7}$$

Here, the distinction should be made that $\mathbf{K}^{(l)}$ is the stiffness matrix of the lth layer and $\mathbf{K}$ is the assembled global stiffness matrix of all layers. Equation (7) is derived for each wave heading angle θ_i, and thus the angle-average must be considered for accounting for the full range of wave propagation directions when deriving the angle averaged damping loss factor as follows:

$$\eta^* = \frac{\int_\Theta \eta(\theta_i) d\theta_i}{\int_\Theta d\theta_i} \tag{8}$$

The variation of the angle θ_i is obtained by rotating the relative orientation of the x-axis with respect to the composite material (Figure A.1). The circular frequency for the wave number k can be written in terms of the Rayleigh quotient as

$$\omega^2 = \frac{\hat{\mathbf{u}}^H \mathbf{K}(k) \hat{\mathbf{u}}}{\hat{\mathbf{u}}^H \mathbf{M} \hat{\mathbf{u}}} \tag{9}$$

Figure A.2 shows the wavenumber of an arbitrary wave propagating in an anisotropic medium as a function of wave heading, θ_i. The distribution and the direction of energy flow in anisotropic media like composites exhibit directional dependence, thus the direction of the energy flow, depicted as θ_e in Figure A.2, is usually different from the wave heading, θ_i. Thus, an appropriate correction must be introduced in the calculation of the group speed.

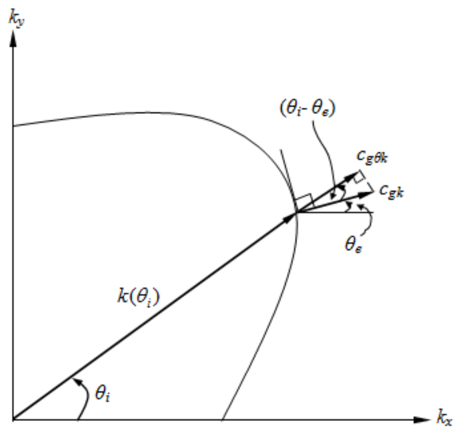


Figure A.2: Wavenumber as a function of wave heading in the k (wave number) plane

The group speed in the direction of wave propagation is $c_{g\theta} = d\omega/dk$, and the differentiation of equation (9) with respect to the wavenumber, k, results in:

$$c_{g\theta} = \frac{\hat{\mathbf{u}}^H(\partial\mathbf{K}(k)/\partial k)\hat{\mathbf{u}}}{2\omega\hat{\mathbf{u}}^H\mathbf{M}\hat{\mathbf{u}}} \tag{10}$$

Referring to Figure A.2, the following relationship holds:

$$c_g = \frac{c_{g\theta}}{\cos(\theta_i - \theta_e)} \tag{11}$$

where the heading of group speed, θ_e may be derived from the geometric interpretation of the wavenumber curve shown in Figure A.2

$$\tan\theta_e = -\frac{\partial k_x/\partial\theta_i}{\partial k_y/\partial\theta_i} = -\frac{(\partial k(\theta_i)/\partial\theta_i)\cos\theta_i - k(\theta_i)\sin\theta_i}{(\partial k(\theta_i)/\partial\theta_i)\sin\theta_i + k(\theta_i)\cos\theta_i} \tag{12}$$

Therefore, the angle-averaged group speed can be evaluated for the full range of wave propagation directions as follows:

$$c_g{}^* = \frac{\int_\Theta c_g(\theta_i)d\theta_i}{\int_\Theta d\theta_i} \tag{13}$$

In the EFEA formulation for composites, c_g^* and η^* from Equations (13) and (8) respectively, are used when deriving the element level EFEA matrices.

Appendix B – Derivation of Power Transfer Coefficients between Composite Members and Derivation of Joint Matrices

The derivation of power transfer coefficients is based on considering a connection between semi-infinite members, prescribing an impinging wave type originating from one of the members, and evaluating all transmitted or reflected wave types in all members. Once available, the power transfer coefficients are used for computing the EFEA joint matrices which connect elements across a discontinuity in the EFEA system of equations. The formulation for computing power transfer coefficients between composite members is presented here. It must be noted, that this formulation accounts for First order Shear Deformation Theory (FSDT) effects, which are important when considering sandwich composite panels. The extended wave dynamic stiffness matrix approach is used to calculate the wave power transfer coefficients of coupled composite members.

Figure B.1. presents a junction consisting of N semi-infinite composite members. The coordinate systems and the translational and rotational displacements are also presented in Figure B. 1.

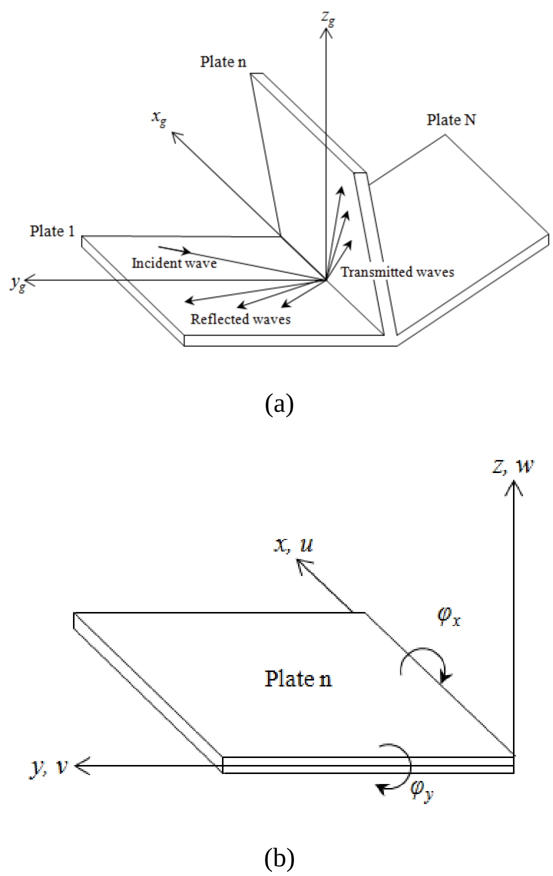


Figure B.1: (a) A general N-plate junction and global coordinate system; (b) local coordinate system and displacements for plate n

For either a laminated or a sandwich composite member, the FSDT allows writing the dynamic equations of motion in the following matrix-vector form:

$$\mathbf{Lu} = \mathbf{0} \tag{1}$$

where $\mathbf{u} = \left[u, v, w, \varphi_x, \varphi_y\right]^T$ and $\mathbf{L}$ is a linear differential operator which can be expressed in terms of elastic constants A_{ij}, B_{ij} and D_{ij} and inertial properties $I_\alpha(\alpha = 0,1,2)$. Substitution of a wave form, $\mathbf{u} = \hat{\mathbf{u}}e^{-ik_x x+\lambda y+i\omega t}$ into equation (1) yields

$$(\lambda^2\mathbf{b}_2 + \lambda\mathbf{b}_1 + \mathbf{b}_0)\hat{\mathbf{u}} = 0 \tag{2}$$

where $\mathbf{b}_0$, $\mathbf{b}_1$, and $\mathbf{b}_2$can be obtained from $\mathbf{L}$ by replacing $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial t}$ by $(-ik_x)$,λ, and ω. This quadratic eigenvalue problem can be solved for five pairs of λ's and $\hat{\mathbf{u}}$'s for given k_x and ω. Then, the edge displacements, $\mathbf{d}_\mathrm{y}$ and the corresponding elastic tractions, $\mathbf{t}_\mathrm{y}$ of each plate can be expressed in terms of the reflected/transmitted wave amplitudes $\mathbf{c} = [c_1, c_2, c_3, c_4, c_5]^\mathbf{T}$ and an incident wave amplitude c_0 as follows

$$\mathbf{d}_\mathrm{y} = [\mathbf{d}_{\mathrm{y}1} \;\; \mathbf{d}_{\mathrm{y}2} \;\; \cdots \;\; \mathbf{d}_{\mathrm{y}5}]\mathbf{c} + \mathbf{d}_{\mathrm{y}0}c_0 \tag{3}$$

$$\mathbf{t}_\mathrm{y} = [\mathbf{t}_{\mathrm{y}1} \;\; \mathbf{t}_{\mathrm{y}2} \;\; \cdots \;\; \mathbf{t}_{\mathrm{y}5}]\mathbf{c} + \mathbf{t}_{\mathrm{y}0}c_0 \tag{4}$$

where $\mathbf{d}_{\mathrm{y}i}$and $\mathbf{t}_{\mathrm{y}i}$(i=1,2,…,5) represent edge displacements and elastic tractions due to the ith transmitted or reflected wave and $\mathbf{d}_{\mathrm{y}0}$ and $\mathbf{t}_{\mathrm{y}0}$ are those for incident waves. Equations (3) and (4) can be combined to give

$$\mathbf{t}_\mathrm{y} = \mathbf{K}\mathbf{d}_\mathrm{y} - \mathbf{f}_{\mathrm{y}0} \tag{5}$$

where $\mathbf{K} = [\mathbf{t}_{\mathrm{y}1} \;\; \mathbf{t}_{\mathrm{y}2} \;\; \cdots \;\; \mathbf{t}_{\mathrm{y}5}][\mathbf{d}_{\mathrm{y}1} \;\; \mathbf{d}_{\mathrm{y}2} \;\; \cdots \;\; \mathbf{d}_{\mathrm{y}5}]^{-1}$ is the wave dynamic stiffness matrix and $\mathbf{f}_{\mathrm{y}0}$= ($\mathbf{Kd}_{\mathrm{y}0}$- $\mathbf{t}_{\mathrm{y}0}$)c_o is the force due to the incident wave. It should be noted that $\mathbf{f}_{\mathrm{y}0} = \mathbf{0}$ in the plates where no incident wave exists. By implying subscript "y" for the remaining discussion, equation (5) can be rewritten for the *n*th plate as

$$\mathbf{t}_\mathrm{n} = \mathbf{K}_\mathrm{n}\mathbf{d}_\mathrm{n} - \mathbf{f}_\mathrm{n} \tag{6}$$

The wave dynamic matrix is assembled to yield an equation for the computation of power transmission coefficients. Since $\mathbf{d}_\mathrm{n}$ and $\mathbf{t}_\mathrm{n}$ are defined in the local coordinate system, they need to be transformed to those at the common junction defined in the global coordinate system, $\mathbf{d}_J$ and $\mathbf{t}_J$ as follows:

$$\mathbf{d}_\mathrm{n} = \mathbf{R}_\mathrm{n}\mathbf{d}_J; \; \mathbf{t}_J = \mathbf{R}_\mathrm{n}^\mathbf{T}\mathbf{t}_\mathrm{n} \tag{7}$$

where $\mathbf{R}_\mathrm{n}$ is a simple coordinate transformation matrix consisting of $\cos\psi_n$ and $\sin\psi_n$. Since the force equilibrium equation for the N plates connected through a structural joint is $\sum_{\mathrm{n}=1}^{\mathrm{N}} \mathbf{R}_n^\mathbf{T}\mathbf{t}_n = \mathbf{0}$, using equations (6) and (7) results in:

$$\left[\sum_{\mathrm{n}=1}^{\mathrm{N}} \mathbf{R}_\mathrm{n}^\mathbf{T}\mathbf{K}_\mathrm{n}\mathbf{R}_\mathrm{n}\right]\mathbf{d}_J = \left[\sum_{\mathrm{n}=1}^{\mathrm{N}} \mathbf{R}_\mathrm{n}^\mathbf{T}\mathbf{f}_\mathrm{n}\right] \tag{8}$$

where $\mathbf{d}_J$ can be evaluated from Equation (8). Using equation (6) the wave amplitudes are evaluated:

$$\mathbf{c} = [\mathbf{d}_{y1} \;\; \mathbf{d}_{y2} \;\; \cdots \;\; \mathbf{d}_{y5}]^{-1}\left(\mathbf{d}_{y} - \mathbf{d}_{y0}c_0\right) \tag{9}$$

Then, the power transfer coefficients can be readily calculated from:

$$\tau_{ij}(\omega,\theta) = \frac{\left|c_j\right|^2}{|c_{0i}|^2} \tag{10}$$

where c_j is the amplitude of a reflected or transmitted wave type j for the given c_{0i}, which is the amplitude of an incident wave type i with frequency ω and an incidence angle of θ.

The above computed transmission coefficients are highly dependent on the angle of incidence, θ for anisotropic media (i.e. composite panels). Thus, the power transfer coefficients are obtained through averaging of the power transmission coefficients with respect to the angle of incidence:

$$\langle\tau_{ij}\rangle = \int_{\Theta} \frac{\tau_{ij}(\theta)c_{gy}(\theta)}{c(\theta)c_{g\theta}(\theta)}d\theta \bigg/ \int_{\Theta} \frac{c_{gy}(\theta)}{c(\theta)c_{g\theta}(\theta)}d\theta \tag{11}$$

The power transfer coefficients are used for creating the EFEA joint matrices which connect elements across a discontinuity. The transmission coefficient associated with each one of the generated waves is calculated as the ratio between the power of the transmitted wave divided by the power of the wave incident to the junction. The complete set of transmission coefficients for the junction is written in the form $\tau_{pr}^{ij}(\omega,\phi)$ where i, p ω, and ϕ represent respectively, the carrier plate, the wave type, the frequency and the heading of the incident wave, and j , r represent respectively the carrier plate and the wave type of the transmitted wave. The matrix of the power transfer coefficients is computed $[\tau]_j^i$. The joint matrices in the EFEA formulation define the power transfer across elements at the joints and are derived from the averaged power transfer coefficients:

$$[J]_j^i = ([I]-[\tau]_j^i)([I]+[\tau]_j^i)^{-1}\int_B \phi_i\phi_j dB \tag{12}$$

where ϕ_i , ϕ_j are Lagrangian basis functions, B is the boundary area between elements i and j at the joint, and $[\tau]_j^i$ is a matrix comprised by the averaged power transfer coefficients through the connection.